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THE LIMITING MOTIONS OF SYSTEMS WITH DRY FRICTION†

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It is shown that in systems with dry friction governed by the Amontons–Coulomb law motions exist in which dissipation of mechanical energy occurs after any time interval however long it may be.

IN MANY cases a system with friction moves in such a manner that the work of the friction forces become smaller and smaller in magnitude [1, p. 124]. Appell has called this property the tendency of material systems to avoid friction and demonstrated it by proving the following statement.

Let the material system under consideration:

(a) be subject to some time-independent constraints,

(b) be subject to the action of internal forces with a potential function which is positive or zero for all possible positions of the system and which vanishes when the system is in a position of stable equilibrium under the action of internal forces only,

(c) contain rigid bodies or points each sliding along the other or along stationary bodies, and

(d) have a force function of external forces that remains less than a certain value for all possible positions of the system in which there is at least one contact producing a force of sliding friction.

The upper limit value of the power of friction forces then vanishes in the time interval in which the system has at least one contact with sliding.

If we accept the classical Amontons–Coulomb law on sliding friction, we can conclude that the normal components of the reaction at some contact points vanish (and it seems that the system tries to be released from frictional constraints at those points). The velocities of sliding decrease in magnitude, and sliding tends to disappear at the other contact points to which sliding friction forces are applied. These conclusions still hold if we accept the general law of friction, namely, when a rigid body A slides along a rigid body B, assumed to be stationary, and the friction force is positive ($F>0$) and has the direction opposite to the velocity $\mathbf{V}$ of the contact point, $F=0$ if and only if the normal reaction is equal to zero. Thus, the power of the friction force being equal to $-F|\mathbf{V}|$ is essentially of negative value which vanishes only when the normal reaction or the velocity of sliding is equal to zero.

But it is not impossible to conclude from the above statement that sliding with friction will terminate after a finite period of time, i.e. sliding with friction vanishing asymptotically in time or a permanent alternation of rolling and sliding will be impossible in a system that satisfies conditions (a)–(d). The problem of the possibility of pure rolling as the steady state of motion is attained in a system for a long but finite lapse of time and is meanwhile of fundamental importance when proving the correctness of the classical model for non-holonomic systems.

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Methods of analytical mechanics enable us to derive the complete system of differential equations for the problem of a rigid body rolling without sliding along the surface of another, and it is not necessary to specify the forces that arise at the point of contact and ensure that there is no relative sliding of the bodies at this point. This system is ideal according to Lagrange, and the method of virtual displacements provides the equations of motion of the bodies in a form that does not contain reaction forces explicitly. Having the equations of motion, one can derive formulae for the reaction forces but of course one cannot determine their origin.

In the general case, the kinematical condition for there to be no sliding at the point of contact of two bodies is defined by non-holonomic equations of the constraints, and the model of the rolling of a body without sliding is an idealization itself because in the large variety of motions of real bodies there are both rolling and sliding, which alternate during a final period of time of an observation. In order to derive the equations of motion of a system with sliding, it is necessary to specify the reaction force that arises at the point of contact of the bodies. As a rule, the reaction is modelled by the friction force (dry, viscous or combined friction) or by the force of creep.

1. We will assume that a force of dry friction acts from a reference rigid body upon another body moving along it. According to the Amontons–Coulomb law, in the case of pure rolling the ratio of the tangential and normal components of the reaction computed using the equations of motion of the body and the equations of the constraints must be less in magnitude than the coefficient of friction at rest. Otherwise, rolling without sliding is impossible, and one cannot use the non-holonomic model.

But this condition is insufficient to enable us to conclude that the model for a system with rolling is correct when using the friction forces: solutions of the non-holonomic model will be adequate to reality only if any small sliding that may be due to reasons not taken into account will vanish in a short period of time. In particular, there should be no vanishing sliding in which zero velocity of sliding is attainable only at separate points of the numerical semi-axis $t \geqslant t_0$.

The sufficient conditions for the absence of vanishing sliding in a mechanical system with dry friction have been given by Pozharitskii [2]. Those conditions are extremely lengthy. For this reason, it seems, there are no applications of the theoretical results obtained in [2].

It turns out that sliding of infinite duration or a permanent alteration of sliding and rolling is quite possible in systems with dry friction. This is illustrated below by the two simplest examples.

2. Consider the material curve along which a heavy bead can move with friction. Let the equation of the curve have the form

$$y = -x^2/2$$

with a suitable choice of the system of coordinates.

The bead can be in equilibrium if $-k \leqslant x \leqslant k$, where $0 < k < 1$ is the coefficient of friction.

We will derive the equations of motion of the bead moving to the point $D(-k, -k^2/2)$ from the right. The natural equations have the form

$$v^2/\rho = P_n - N, \quad dv/dt = P_\tau - kN$$

Here v is the modulus of the velocity of the bead, ρ is the radius of curvature, P_τ and P_n are the projections of the gravity force onto the tangential and normal directions, and N is the normal pressure acting from the curve on the bead. The mass of the bead and its weight are assumed to be unity. These equations can be written in the form

$$dx/dt = vr^{-1}, \quad dv/dt = [r^2(k - y') + ky''v^2]\, r^{-3}; \quad r = (1 + y'^2)^{1/2}$$

(the prime denotes the derivative with respect to x), whence we find

$$du/dx = k - y' + 2ky''ur^{-2}, \quad u = v^2/2$$

The general solution of this linear equation has the form

$$u(x) = [u(-1) + \int_{-1}^{x} \frac{k-y'}{E(w)} dw]\, E(x), \quad E(x) = \exp\left\{\int_{-1}^{x} \frac{2dy''}{r^2} dx\right\}$$

When

$$u(0) = \int_{-k}^{0} \frac{k-y'}{E(w)} dw$$

we have

$$u(-k) = 0, \quad \frac{du}{dx}(-k) = 0.$$

Therefore, the limit

$$\lim_{x \to -k} \int_{-1}^{x} \frac{rdx}{u(x)^{1/2}} = +\infty$$

holds for this solution, i.e. while the bead moves under the specified initial conditions it reaches the point D after an infinite time.

3. Consider the plane motion of a heavy inhomogeneous wheel along a rectilinear rail.

The equations of motion. We will write the equations of motion of the wheel with sliding [2, 3] in the dimensionless form

$$\dot\varphi = \omega, \quad \dot\omega = P\omega^2 + Q, \quad \dot v = R\omega^2 + W \tag{3.1}$$
$$P = (\eta - 1)j/s, \quad Q = -j/s, \quad j = \xi + \eta\mu k, \quad \sigma = \rho/r$$
$$R = \xi + (\eta - 1)(j\eta + \lambda\mu k)/s, \quad W = -(j\eta + \lambda\mu k)/s$$
$$\xi = \sigma\sin\varphi, \quad \eta = 1 - \sigma\cos\varphi, \quad s = \lambda + \xi j, \quad \mu = \operatorname{sgn} v$$
$$\lambda = \kappa^2/r^2, \quad \omega = \Omega/l, \quad v = V/(rl), \quad \tau = lt, \quad l = (g/r)^{1/2}$$

Here the phase variables φ, Ω and V are, respectively, the angle of rotation, the angular velocity of the wheel, and the linear velocity of its point touching the rail. There are also the following parameters: r is the geometrical radius of the wheel, $\varkappa$ is the central radius of inertia, ρ is the offset of the centre of mass, g is the acceleration due to gravity, and k is the coefficient of friction. The dot denotes the derivative with respect to the dimensionless time τ.

The functions ξ and η are the coordinates of the centre of mass C with respect to the orthogonal axes Pxy performing the translational motion (P is the point of contact of the wheel and the rail, and the axis Py is directed vertically upwards), and φ is the angle between the descending vertical and the segment connecting the geometrical centre O of the wheel and the point C.

In the case of pure rolling, the equations of motion of the wheel have the form

$$\dot{\varphi} = \omega, \quad \dot{\omega} = H(\omega^2 + 1); \quad H = -\xi(\xi^2 + \eta^2 + \lambda)^{-1/2} \tag{3.2}$$

The mechanical system under consideration has a variable structure, namely, rolling and sliding can alternate, and wheel detachment from the rail is possible at high angular velocity.

Conditions for the change in the patterns of motion. For the argument that follows it is necessary to establish the conditions for a transition to occur from sliding to rolling and vice versa.

Incidentally, note that

$$(P\omega^2 + Q)\,|_{v=0} \neq H(\omega^2 + 1)$$

which is typical for systems with dry friction.

If $v=0$ at a certain instant of time rolling will occur subsequently under the conditions that at this instant the inequalities

$$\left| \frac{[\omega^2(\xi^2 + \lambda) + \nu\eta]\,\xi}{\omega^2\xi^2\eta + \nu(\eta^2 + \kappa^2)} \right| < k \tag{3.3}$$

and $s>0$ are satisfied and the constraint is in tension, i.e.

$$\omega^2\xi^2\eta + \nu(\eta^2 + \kappa^2) < 0 \quad (\nu = \omega^2(\eta - 1) - 1) \tag{3.4}$$

The conditions for the transition from rolling to sliding to occur in the general case of the plane motion of a rigid body along an arbitrary curve with dry friction can be found in [4] where an elegant geometrical interpretation of these conditions is also given. The inequalities (3.3) and $s>0$ (see below) encapsulate them for the system considered.

One can obtain condition (3.4) in the following way. Let us release the wheel from the constraint and change the action of the latter by the reaction force. The theorem on the motion of the centre of mass C yields $m\ddot{y}_C = N - mg$ as the projection onto the axis Py whence it follows that $\ddot{y}_C + g > 0$ if the constraint is in tension. But we then have $y_C = r - \rho\cos\varphi$. Hence, the required condition takes the form

$$\sigma(\dot{\omega}\sin\varphi + \omega^2\cos\varphi) + 1 > 0.$$

If we substitute the expression for $\dot{\omega}$ from Eqs (3.2) into this inequality we obtain (3.4). By substituting $\dot{\omega}$ from (3.1) we obtain the condition of tension of the constraint with the wheel slides

$$\nu < 0 \tag{3.5}$$

Taking conditions (3.4) into account we can write inequality (3.3) in the form

$$|I| + Jk < 0; \quad I = \left[\omega^2 \frac{m+\lambda}{\eta} - \eta\right]\xi, \quad J = \omega^2 m - \lambda - \eta^2 \tag{3.6}$$

$$m = \eta^2 + \eta(\lambda + \sigma^2 - 1) - \lambda$$

The further analysis will be carried out under certain quite natural restrictions on the parameters. We will assume that the centre of mass is located inside the wheel (i.e. $\sigma<1$), the square of the dimensionless central radius of inertia satisfies the inequality $\lambda > 1/4$ (if this condition holds, the function $i = \sigma^2 - \sigma\cos\varphi + \lambda$, which occurs frequently in the discussion that follows, is positive for all values of φ), the coefficient of friction lies in the range $0<k<1$, and the values of the parameters λ, σ and k ensure the validity of the condition $s(\varphi)>0$ for all values $0 \leqslant \varphi \leqslant 2\pi$ and $\mu = \pm 1$. It is well known [4] that when $s < 0$ a shock reaction can arise in the system and, in the case of a bilateral constraint, the non-uniqueness of the solutions (Painlevé's paradox) can occur.

Remove the modulus sign in inequality (3.6) if $\xi \geqslant 0$, i.e. if $0 \leqslant \varphi \leqslant \pi$. When $\pi \leqslant \varphi \leqslant 2\pi$ the situation is completely symmetrical.

In the specified range of variation of the angle φ the function $m(\varphi)$ has a single zero $0 < \varphi^* < \pi/2$ since $m(0) < 0$, $m(\pi/2) > 0$, $\eta > 0$. The coefficient of ω^2 in the expression for I is positive. Therefore

$$I < 0 \text{ for } \omega^2 < \bar{\omega}^2, \; I > 0 \text{ for } \omega^2 > \bar{\omega}^2; \; \bar{\omega}^2 = \eta^2/(m+\lambda)$$

We have

$$I + Jk = s_+(R_+\omega^2 + W_+), \quad -I + Jk = -s_-(R_-\omega^2 + W_-) \tag{3.7}$$

The values of the corresponding expressions computed when $\mu = 1$ or $\mu = -1$ are marked by the plus or minus subscripts.

Let us find the ranges of constant sign of the function

$$F_+(\xi) = \frac{-W_+}{R_+} = \frac{j_+\eta + \lambda\eta k}{j_+m + \lambda\xi}$$

Its numerator does not vanish ($\xi > 0$). The denominator $Z(\varphi)$ has exactly one zero $\varphi' \in (0, \varphi^*)$. In fact, we have

$$Z(0) = m(0)k < 0, \quad Z(\varphi^*) = \lambda\sigma\sin\varphi^*/(1 - \sigma\cos\varphi^*) > 0$$
$$dZ/d\varphi = j_+\xi + (1 - \eta + \xi k)i > 0, \quad \varphi \in (0, \varphi^*)$$
$$Z(\varphi) > 0 \text{ for } \varphi^* < \varphi \leqslant \pi$$

Thus, the inequality $I + Jk < 0$ holds for $0 < \varphi < \varphi'$ and for any values of ω^2. When $\pi \geqslant \varphi > \varphi'$ we have

$$I + Jk > 0, \text{ if } \omega^2 > F_+(\xi)$$

$$I + Jk < 0, \text{ if } \omega^2 < F_+(\xi)$$

In a similar manner we investigate the function

$$F_-(\xi) = \frac{-W_-}{R_-} = \frac{j_-\eta - \lambda k}{\lambda k + j_- i}$$

The zeros of its numerator N correspond to the points of intersection of the branch of the hyperbola $\xi = (\eta + \lambda/\eta)k$ and the circle $\xi^2 + (\eta - 1)^2 = \sigma^2$. There are two possible points of intersection. We will not discuss the case of tangency because it is not typical. It is obvious that the curves are disjoint provided that

$$\sigma < 2k\sqrt{\lambda} \tag{3.8}$$

Let us denote the zeros of the function N by $\varphi_1 < \varphi_2$.

The denominator $D(\varphi) > 0$ at $\varphi \in [0, \bar{\varphi}]$, where $\bar{\varphi} < \pi$ is the angle for which $j = 0$. On the other hand

$$D(\pi) = -m(\pi)k < 0, \quad dD/d\varphi = \xi j_- + (1-\eta)i < 0, \quad \varphi \in (\bar{\varphi}, \pi)$$

Hence, a single zero φ'' of the function D exists in the interval $(\bar{\varphi}, \pi)$, and we have $0 < \varphi_1 < \varphi_2 < \varphi'' < \pi$ since $N < 0$ in the interval specified.

Thus, if N is not equal to zero, then $-I + Jk < 0$ for $0 < \varphi < \varphi''$ and for any values of ω^2. In the case when $N(\varphi_1) = N(\varphi_2) = 0$, this inequality is satisfied in the interval $(0, \varphi_1) \cup (\varphi_2, \varphi'')$ and for any values of ω^2.

It can be verified that

$$\bar{\omega}^2 < F_-(\xi), \quad \pi > \varphi > \varphi''; \quad \bar{\omega}^2 < F_+(\xi), \quad \pi > \varphi > \varphi'$$
$$F_+(\xi) < (\eta - 1)^{-1}, \quad \pi > \varphi > \pi/2$$

Let us summarize the results of the investigation of the conditions wherein sliding of the wheel occurs. We will assume that $v = 0$ at this instant of time.

(a) If the offset of the centre of mass of the wheel from its geometrical centre is small so that inequality (3.8) is satisfied, then rolling of the wheel will be preserved for $\omega^2 < \bar{\omega}^2$ and also for $\omega^2 > \bar{\omega}^2$ but with $\varphi \in [0, \varphi']$. When $\varphi' < \varphi < \pi$ and $\bar{\omega}^2 < \omega^2 < F_+$ sliding will not occur but when $\omega^2 > F_+$ it will start when $v = 0$ (by virtue of relations (3.1), (3.7), $\operatorname{sgn} v = \operatorname{sgn} I$). If $\pi/2 < \varphi < \pi$, then condition (3.5) for the constraint to be in tension must be satisfied.

(b) Let $\omega^2 < \bar{\omega}^2$. If the function F_- vanishes at two points of the interval $(0, \varphi'')$ that occurs for a wheel with a large offset of the centre of mass, then rolling will be sustained when $\varphi \in [0, \varphi'] \cup [\varphi_2, \pi]$. Sliding with $v < 0$ will inevitably start in the interval $\varphi_1 < \varphi < \varphi_2$. When $\omega^2 > \bar{\omega}^2$ the conditions for the transition from rolling to sliding will remain the same as in case (a).

It is worth noting that in case (b) when the value of the angular velocity is small enough the absence of sliding cannot be ensured in principle by the friction force.

The inequalities were analysed for $\xi > 0$. The substitution of $-\xi$ for ξ yields a situation which is symmetrical to that considered above. One need only take into account that $F_-(\xi) = F_+(-\xi)$.

Stability of rolling. The correctness of the modelling of the interaction force between the wheel and the rail by the force of dry friction means, in particular, that if, for some reason that was ignored when choosing the initial conditions for rolling, the velocity of sliding is not zero but small in magnitude, sliding motion will disappear and will transform into rolling in a short time. This requirement is quite natural and follows from the fact that the accuracy of measurements with physical instruments is limited and the reality is only roughly approximated by mathematical models.

We will consider a wheel with a small offset of the centre of mass (condition (3.8) is satisfied). Let $v = 0$ at the initial instant. Sliding will not occur if

$$\varphi \in [-\varphi', \varphi'] \quad \text{or} \quad \omega^2 < F - \epsilon \tag{3.9}$$
$$F = \begin{cases} F_+(\xi), & \text{if} \quad \varphi' < \varphi \leqslant \pi \\ F_-(\xi), & \text{if} \quad \pi < \varphi < 2\pi - \varphi' \end{cases}$$

where the constant $\epsilon > 0$ is chosen in such a manner that $F - \epsilon > 0$.

The total mechanical energy of the wheel is constant while the wheel is rolling. Therefore

$$\frac{1}{2}(\lambda + \xi^2 + \eta^2)\omega^2 + \eta = h$$

Using condition (3.9) we find that

$$h < \frac{1}{2}(\lambda + \xi^2 + \eta^2)(F - \epsilon) + \eta$$

in the interval $(\varphi', 2\pi - \varphi')$.

The right-hand side of this inequality is a positive function of φ when $\epsilon \ll 1$, it is unbounded at the ends of the interval specified, and a minimum $h(\epsilon)$ of this function exists.

Assume that $h < h(\epsilon)$ and condition (3.9) is satisfied at the initial instant. Then sliding will never occur. One can prove that if sliding existed at the initial instant it will cease after a finite time.

It is sufficient to consider the case $\xi > 0$. Let $v = 0$. When $0 \leqslant \varphi \leqslant \varphi'$ we have

$$\dot{v} = R_+\omega^2 + W_+ < W_+ \leqslant -m_1 < 0, \quad m_1 = \min\left(\frac{j_+\eta + \lambda k}{s_+}\right)$$

Because of the continuity this negative-valued upper bound holds in the half-interval $[\varphi', \varphi' + \delta_1)$ if $\delta_1 > 0$ is sufficiently small. When $\varphi' + \delta_1 \leqslant \varphi \leqslant \pi$ we have

$$\dot{v} = R_+(\omega^2 - \dot{F}_+) < -\epsilon R_+ \leqslant -\epsilon m_2 < 0, \quad m_2 = \min\left(\frac{j_+ m(\varphi) + \lambda\xi}{s_+\eta}\right)$$

Let $v < 0$. If $0 \leqslant \varphi \leqslant \varphi''$, then

$$\dot{v} = R_-\omega^2 + W_- \geqslant W_- > k(\eta - \sqrt{\lambda})^2/s_-$$

by virtue of (3.8). The estimate is true in a small half-interval $[\varphi'', \varphi'' + \delta_2)$, $\delta_2 > 0$. When $\varphi'' + \delta_2 \leqslant \varphi \leqslant \pi$ we have

$$\dot{v} = R_-(\omega^2 - F_-) > -\epsilon R_- > 0$$

since $F_+ \leqslant F_-$ and $R_- < 0$ over the interval specified.

Thus, the sliding that occurred will cease after a finite time.

Motion with constant alternation between rolling and sliding. Put $h(0) = h_*$. Let $\varphi = v = 0$, $\omega > 0$ and the constant $h = h_0 > h_*$ at an initial instant. If $\omega^2 < F$, the wheel will roll without sliding until the point which represents the motion of the system in the (h, φ) plane intersects the graph of the function

$$G(\varphi) = \frac{1}{2}(\lambda + \xi^2 + \eta^2)F + \eta$$

at $\varphi = \varphi_0$ moving initially along the straight line $h = h_0$. This function is even with respect to φ and attains a minimum value $h_* > 0$ at the two points $\varphi' < \varphi_m < \pi$ and $2\pi - \varphi_m$.

When $\varphi > \varphi_0$ sliding will start because the inequality $h_0 > G$ implies that $\omega^2 > F$. The graph of the function F in the (ω^2, φ) plane is qualitatively similar to that of $G(\varphi)$.

When $\varphi \leqslant \varphi_0$ the representative point moves along an integral curve of the equation

$$d\omega^2/d\varphi = 2H(\omega^2 + 1) \tag{3.10}$$

and, when $\varphi > \varphi_0$, of the equation

$$d\omega^2/d\varphi = 2(P_+\omega^2 + Q_+)$$

until sliding ceases. In the case $h_0 = h_*$, the corresponding integral curve Γ of Eq. (3.10) touches the graph of $F(\varphi)$ at the point φ_m which is not, generally speaking, the point of a minimum of this function. In the general case, the curvatures of the curves Γ and F are distinct at the point

$\varphi=\varphi_m$, i.e.

$$\left.\frac{d^2(\omega^2-F)}{d\varphi^2}\right|_{\varphi m}=\left.\left(4H^2(\omega^2+1)+2(\omega^2+1)\frac{dH}{d\varphi}-\frac{d^2F}{d\varphi^2}\right)\right|_{\varphi m}\neq 0. \qquad (3.11)$$

and the curvature of the curve G does not equal zero.

When the values of $\Delta h=h_0-h_*$ are small the sliding motion of the wheel is split into two sections. In the "acceleration" section the velocity v increases monotonically from zero to

$$v_{max}=\frac{1}{2}c(\varphi_0)(\varphi_{01}-\varphi_0)^2+\dots,\quad c(\varphi)=\frac{d}{d\varphi}\left(\frac{R_+(\omega^2-F)}{\omega}\right)$$

In the "deceleration" section the velocity v of sliding decreases from v_{max} to zero, and where

$$-v_{max}=\frac{1}{2}c(\varphi_{01})(\varphi_{02}-\varphi_{01})^2+\dots$$

Here $\varphi_{01}>\varphi_0$ is the value of φ at which the representative point leaves the region enclosed by the curve F ($v_{max}=v(\varphi_{01})$) according to the third equation of system (3.1), φ_{02} is the angle at which a new rolling starts, and the dots denote infinitesimal terms of the third order and higher in the Taylor expansions.

Note that the transition from rolling to sliding occurs without singularities, while in the opposite transition the acceleration $\dot{v}$ has a discontinuity (the so-called "soft" shock).

By virtue of (3.11), the smallness of Δh, and the continuity of the functions $\omega(\varphi)$, $G(\varphi)$ and $F(\varphi)$, the values of $\Delta\varphi=\varphi_{02}-\varphi_0$ and $\varphi_m-\varphi_0$ are the equivalent infinitely small quantities ($\Delta\varphi\sim\varphi_m-\varphi_0$), and

$$\Delta h\sim(\Delta\varphi)^2 \qquad (3.12)$$

(estimate (3.12) may be obtained by replacing the curve G by an osculating parabola in a small neighbourhood of the point $\varphi=\varphi_m$ in the (h,φ) plane).

When the wheel slides its total mechanical energy (apart from a multiplier and an additive constant) is equal to

$$E=\frac{1}{2}[(v-\omega\eta)^2+(\xi^2+\lambda)\omega^2]+\eta$$

The power of the friction force is $\dot{E}=v\mu\lambda kv/s<0$.

Since $c(\varphi_m)|_T=0$, we have $c(\varphi_{01})\sim\Delta\varphi$ and $c(\varphi_{02})\sim\Delta\varphi$. Hence, $v_{max}\sim(\Delta\varphi)^3$, and the energy loss for one cycle of sliding is

$$\Delta E\sim(\Delta\varphi)^4 \qquad (3.13)$$

From estimates (3.12) and (3.13) it follows that the total mechanical energy of the wheel satisfies the condition $h_1>h_*$ after sliding has terminated. Because of this, the short-term sliding near the points $\varphi=2\pi-\varphi_m$ and $\varphi=\varphi_m$ will start no matter what kind of motion, rotational or oscillatory, will thereafter occur. After the ith cycle of sliding the energy is $h_i>h_*$ (the monotonically decreasing sequence $\{h_i\}$ has the value h_* as the limit). Rolling and sliding of the wheel will alternate constantly. The period of sliding then decreases.

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